Variational principles for the static electric and magnetic polarizabilities of anisotropic media with perfect electric conductor inclusions

2009 J. Phys. A: Math. Theor. 42 335403

(http://iopscience.iop.org/1751-8121/42/33/335403)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 335403 (14pp) doi:10.1088/1751-8113/42/33/335403

Variational principles for the static electric and magnetic polarizabilities of anisotropic media with perfect electric conductor inclusions

Daniel Sjöberg

Department of Electrical and Information Technology, Faculty of Engineering, Lund University, Sweden

E-mail: daniel.sjoberg@eit.lth.se

Received 5 April 2009, in final form 22 June 2009
Published 30 July 2009
Online at stacks.iop.org/JPhysA/42/335403

Abstract

We present four variational principles for the electric and magnetic polarizabilities for a structure consisting of anisotropic media with perfect electric conductor (PEC) inclusions. From these principles, we derive monotonicity results and upper and lower bounds on the electric and magnetic polarizabilities. When computing the polarizabilities numerically, the bounds can be used as error bounds. The variational principles demonstrate important differences between electrostatics and magnetostatics when PEC bodies are present.

PACS numbers: 41.20.Cv, 41.20.Gz, 03.50.De, 11.80.Fv

1. Introduction

Variational principles can be viewed as a physical way of interpreting mathematical equations. Instead of giving the relevant physical law as, for instance, a partial differential equation, a variational principle typically defines an energy functional, where the correct physical behavior is obtained for the trial function giving the minimum value of the functional. Typically, the functional can be interpreted as the energy of the system.

In our case, we are interested in calculating the electric and magnetic polarizabilities of a system consisting of anisotropic permittivity and permeability, possibly containing inclusions of metal modeled as a perfect electric conductor (PEC). This can be used in various applications, for instance Rayleigh scattering [1–4], where the size of the scatterer is small compared to the wavelength and hence the induced electric and magnetic dipole moments give the main contributions to the scattered wave amplitude. Another important case is homogenization theory for composite materials [5–7], where it is assumed that all inhomogeneities in a material appear on a scale much smaller than the wavelength. The result is that the effective material properties, which are observable on the wavelength scale, can be determined from the polarizability per unit volume. As a final example of applications, there has recently been a series of publications proposing the use of electric and magnetic polarizabilities to give physical bounds on electromagnetic interaction over all frequencies for antennas, materials and general scatterers [8–12]. For instance, in the case of scattering theory, the bounds state that the extinction cross section of any scatterer integrated over all wavelengths is bounded by a constant multiplied by the sum of the electric and magnetic polarizabilities of the scatterer, no matter how complicated the geometry or material of the scatterer is [8]. In the antenna case, essentially the same result holds if the extinction cross section is replaced by the partial realized gain of the antenna [9]. The common factor for all the applications mentioned is that important physical properties depend directly on the polarizabilities of the system. The results in this paper demonstrate how these numbers can be calculated or bounded using variational principles.

Electrostatics is one of the prime examples of the Laplace equation, and has thus been studied thoroughly. Magnetostatics is somewhat younger, but due to its large financial impact on, for instance, power transformers and hard disk drives, it has also received significant attention [13–15]. A classical problem directly linked to ours is to compute low frequency circuit parameters such as capacitance and inductance [16–19]. It is interesting to note that the problem of magnetic polarizability of a PEC body is mathematically equivalent to the problem of computing the virtual mass [20, p 31] of a body in a uniformly flowing fluid [21–23].

Even though variational principles are typically associated with a self-adjoint operator, we note that in some cases there exist techniques for reformulating the problem so that variational principles can be found for, for instance, complex-valued non-Hermitian matrices describing material properties [24]. However, in this paper we assume that all material properties can be modeled using symmetric, real-valued matrices.

This paper is organized as follows. In section 2, we state the geometry of our problem. In section 3 we summarize the variational principles, two of which are explicitly verified in appendix A. In section 4, it is shown that these principles imply monotonicity results for the polarizabilities. The variational principles are interpreted as giving upper and lower bounds for the polarizabilities in section 5, and a numerical illustration is given in section 6. Finally, some conclusions are given in section 7.

2. Geometry and statement of the problem

We consider the situation of a structure as in figure 1 with anisotropic permittivity and permeability matrices $\boldsymbol{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{\mu}(\boldsymbol{x})$ and possibly PEC inclusions in a region Ω with volume V_Ω, such that $\mathbb{R}^3 \setminus \Omega$ is simply connected. The structure is surrounded by vacuum with permittivity $\boldsymbol{\epsilon}_0 = \epsilon_0 \mathbf{I}$ and permeability $\boldsymbol{\mu}_0 = \mu_0 \mathbf{I}$. The structure is subjected to a homogeneous electric field and a homogeneous magnetic field. The induced redistribution of charges and currents in the structure gives rise to an electric and magnetic dipole moment according to (where $\hat{\boldsymbol{n}}$ is the outward normal vector)

$$\boldsymbol{p} = \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0)\boldsymbol{E}\,\mathrm{d}V + \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}} \cdot \boldsymbol{D}\,\mathrm{d}S \tag{1}$$

$$\boldsymbol{m} = \int_{\mathbb{R}^3\setminus\Omega} \left(\boldsymbol{\mu}_0^{-1} - \boldsymbol{\mu}^{-1}\right)\boldsymbol{B}\,\mathrm{d}V + \frac{1}{2}\oint_{\partial\Omega} \boldsymbol{x} \times (\hat{\boldsymbol{n}} \times \boldsymbol{H})\,\mathrm{d}S, \tag{2}$$

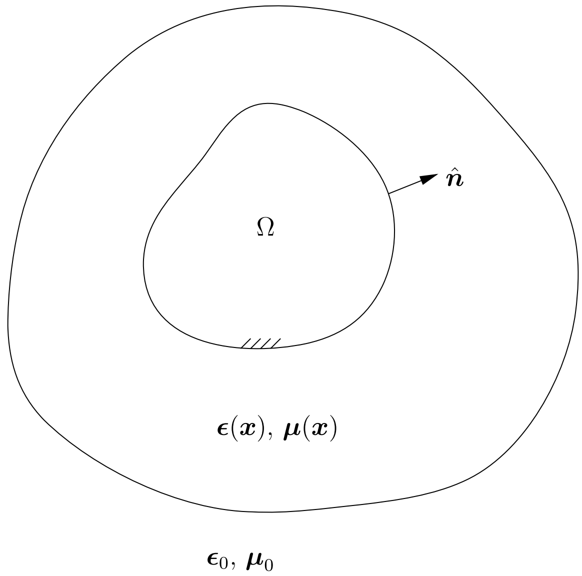

Figure 1. Typical geometry of the structure considered.

where $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are the full electric field strength, electric flux density, magnetic flux density and magnetic field strength, respectively. The electric and magnetic polarizabilities are defined by

$$\boldsymbol{p} = \epsilon_0 \gamma_\mathrm{e} \boldsymbol{E}_0 = \gamma_\mathrm{e} \boldsymbol{D}_0 \tag{3}$$

$$\boldsymbol{m} = \mu_0^{-1} \gamma_\mathrm{m} \boldsymbol{B}_0 = \gamma_\mathrm{m} \boldsymbol{H}_0, \tag{4}$$

where we used the fact that in the surrounding medium, we cannot distinguish between the applied electric field strength $\boldsymbol{E}_0$ and the flux density $\boldsymbol{D}_0$, since these are related by $\boldsymbol{D}_0 = \epsilon_0 \boldsymbol{E}_0$. The same reasoning applies for the magnetic fields.

Even though the problem is primarily stated for finite structures in a three-dimensional space, the final formulation of the variational principles also admits periodic solutions, where for instance the finite structure in figure 1 is repeated periodically without the PEC portion of the copies touching each other. The only thing needed is a natural reformulation of the function spaces.

3. Summary of variational principles

The variational principles can be derived from the static Maxwell's equations by representing the fields using either scalar or vector potentials and constructing natural quadratic forms. Starting from the variational principles themselves, we show in appendix A how it can be verified that the minimizing potentials satisfy the static Maxwell's equations. The variational principles along with the associated classes of admissible potentials are as follows. Using scalar potential φ for the electric field,

$$\begin{aligned} J_\mathrm{e}(\varphi, \boldsymbol{E}_0) = {} & \int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi \cdot \boldsymbol{\epsilon} \nabla\varphi \,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0)\boldsymbol{E}_0 \,\mathrm{d}V \\ & + \boldsymbol{E}_0 \cdot \left[\int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0)\,\mathrm{d}V + V_\Omega \boldsymbol{\epsilon}_0 \right] \cdot \boldsymbol{E}_0 \end{aligned} \tag{5}$$

$$\mathcal{A}_\varphi = \{\varphi \in H_1(\mathbb{R}^3 \setminus \Omega);\ \hat{\boldsymbol{n}} \times (\boldsymbol{E}_0 - \nabla\varphi) = \boldsymbol{0} \text{ on } \partial\Omega\}. \tag{6}$$

Using vector potential $\boldsymbol{F}$ for the electric field,

$$K_{\mathrm{e}}(\boldsymbol{F},\boldsymbol{D}_0)=\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}\nabla\times\boldsymbol{F}\,\mathrm{d}V-2\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F})\cdot\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\boldsymbol{D}_0\,\mathrm{d}V$$
$$-2\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})\,\mathrm{d}S+\boldsymbol{D}_0\cdot\left[-\int_{\mathbb{R}^3\setminus\Omega}\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\mathrm{d}V+V_\Omega\boldsymbol{\epsilon}_0^{-1}\right]\boldsymbol{D}_0 \quad (7)$$

$$\mathcal{A}_{\boldsymbol{F}}=\{\boldsymbol{F}\in H_1(\mathrm{curl},\mathbb{R}^3\setminus\Omega);\ \hat{\boldsymbol{n}}\times\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})=\boldsymbol{0}\text{ on }\partial\Omega\}. \quad (8)$$

Using scalar potential ψ for the magnetic field,

$$J_{\mathrm{m}}(\psi,\boldsymbol{H}_0)=\int_{\mathbb{R}^3\setminus\Omega}\nabla\psi\cdot\boldsymbol{\mu}\nabla\psi\,\mathrm{d}V-2\int_{\mathbb{R}^3\setminus\Omega}\nabla\psi\cdot(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\boldsymbol{H}_0\,\mathrm{d}V$$
$$+2\boldsymbol{H}_0\cdot\frac{\boldsymbol{\mu}_0}{2}\oint_{\partial\Omega}\boldsymbol{x}\times(\hat{\boldsymbol{n}}\times(\boldsymbol{H}_0-\nabla\psi))\,\mathrm{d}S+\boldsymbol{H}_0\cdot\left[\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\,\mathrm{d}V+V_\Omega\boldsymbol{\mu}_0\right]\boldsymbol{H}_0 \quad (9)$$

$$\mathcal{A}_\psi=\{\psi\in H_1(\mathbb{R}^3\setminus\Omega);\ \hat{\boldsymbol{n}}\cdot\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi)=0\text{ on }\partial\Omega\}. \quad (10)$$

Using vector potential $\boldsymbol{A}$ for the magnetic field,

$$K_{\mathrm{m}}(\boldsymbol{A},\boldsymbol{B}_0)=\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{A})\cdot\boldsymbol{\mu}^{-1}\nabla\times\boldsymbol{A}\,\mathrm{d}V-2\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{A})\cdot\left(\boldsymbol{\mu}_0^{-1}-\boldsymbol{\mu}^{-1}\right)\boldsymbol{B}_0\,\mathrm{d}V$$
$$+\boldsymbol{B}_0\cdot\left[-\int_{\mathbb{R}^3\setminus\Omega}\left(\boldsymbol{\mu}_0^{-1}-\boldsymbol{\mu}^{-1}\right)\mathrm{d}V+V_\Omega\boldsymbol{\mu}_0^{-1}\right]\boldsymbol{B}_0 \quad (11)$$

$$\mathcal{A}_{\boldsymbol{A}}=\{\boldsymbol{A}\in H_1(\mathrm{curl},\mathbb{R}^3\setminus\Omega);\ \hat{\boldsymbol{n}}\cdot(\boldsymbol{B}_0+\nabla\times\boldsymbol{A})=0\text{ on }\partial\Omega\}. \quad (12)$$

The minimum values of these functionals are given by

$$\min_{\varphi\in\boldsymbol{A}_\varphi}J_{\mathrm{e}}(\varphi,\boldsymbol{E}_0)=J_{\mathrm{e}}(\varphi_0,\boldsymbol{E}_0)=\boldsymbol{E}_0\cdot\boldsymbol{p}=\epsilon_0\boldsymbol{E}_0\cdot\gamma_{\mathrm{e}}\boldsymbol{E}_0 \quad (13)$$

$$\min_{\boldsymbol{F}\in\boldsymbol{A}_{\boldsymbol{F}}}K_{\mathrm{e}}(\boldsymbol{F},\boldsymbol{D}_0)=K_{\mathrm{e}}(\boldsymbol{F}_0,\boldsymbol{D}_0)=-\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{p}=-\epsilon_0^{-1}\boldsymbol{D}_0\cdot\gamma_{\mathrm{e}}\boldsymbol{D}_0 \quad (14)$$

$$\min_{\psi\in\boldsymbol{A}_\psi}J_{\mathrm{m}}(\psi,\boldsymbol{H}_0)=J_{\mathrm{m}}(\psi_0,\boldsymbol{H}_0)=\boldsymbol{H}_0\cdot\boldsymbol{\mu}_0\boldsymbol{m}=\mu_0\boldsymbol{H}_0\cdot\gamma_{\mathrm{m}}\boldsymbol{H}_0 \quad (15)$$

$$\min_{\boldsymbol{A}\in\boldsymbol{A}_{\boldsymbol{A}}}K_{\mathrm{m}}(\boldsymbol{A},\boldsymbol{B}_0)=K_{\mathrm{m}}(\boldsymbol{A}_0,\boldsymbol{B}_0)=-\boldsymbol{B}_0\cdot\boldsymbol{m}=-\mu_0^{-1}\boldsymbol{B}_0\cdot\gamma_{\mathrm{m}}\boldsymbol{B}_0, \quad (16)$$

where the minimizing potentials $(\varphi_0,\boldsymbol{F}_0,\psi_0,\boldsymbol{A}_0)$ satisfy the electrostatic and magnetostatic equations

$$\nabla\cdot\boldsymbol{D}=\nabla\cdot[\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi_0)]=0 \quad (17)$$

$$\nabla\times\boldsymbol{E}=\nabla\times[\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)]=\boldsymbol{0} \quad (18)$$

$$\nabla\cdot\boldsymbol{B}=\nabla\cdot[\boldsymbol{\mu}(\boldsymbol{H}_0-\nabla\psi_0)]=0 \quad (19)$$

$$\nabla\times\boldsymbol{H}=\nabla\times[\boldsymbol{\mu}^{-1}(\boldsymbol{B}_0+\nabla\times\boldsymbol{A}_0)]=\boldsymbol{0}. \quad (20)$$

We call $J_{\mathrm{e}}(\varphi,\boldsymbol{E}_0)$ and $K_{\mathrm{m}}(\boldsymbol{A},\boldsymbol{B}_0)$ the direct functionals, since they are associated with the natural potentials for electric and magnetic fields. For these potentials, the boundary values can be written in terms of the exciting fields as $\varphi=\boldsymbol{E}_0\cdot\boldsymbol{x}$ (possibly plus a constant scalar) and $\boldsymbol{A}=\frac{1}{2}\boldsymbol{x}\times\boldsymbol{B}_0$ (possibly plus a constant vector). The boundary values of the potentials in the functionals $K_{\mathrm{e}}(\boldsymbol{F},\boldsymbol{D}_0)$ and $J_{\mathrm{m}}(\psi,\boldsymbol{H}_0)$ cannot as easily be expressed in terms of the exciting fields, and we call them the dual functionals. In principle, the direct functionals can be seen as problems with Dirichlet boundary conditions and the dual functionals as problems with Neumann boundary conditions.

An interesting difference between the direct functionals and the dual functionals is that the dual functionals include a term which is the scalar product of the applied field and (twice) the induced dipole moment in the PEC body. In the electric case this term is minimized when the dipole moment is parallel with the applied field, whereas in the magnetic case the term is minimized when the dipole moment is antiparallel to the applied field. This expresses a fundamental difference in sign for the PEC polarizability for electric and magnetic fields. This is further explored in the following section.

4. Monotonicity of the polarizabilities

Consider a situation with two different PEC bodies, Ω and Ω', where $\Omega' \subseteq \Omega$ and $\delta\Omega = \Omega\backslash\Omega'$. Associated with Ω and Ω' are also the permittivity functions $\boldsymbol{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{\epsilon}'(\boldsymbol{x})$ and permeability functions $\boldsymbol{\mu}(\boldsymbol{x})$ and $\boldsymbol{\mu}'(\boldsymbol{x})$, respectively. The spaces of admissible test functions are slightly different, since the boundary conditions are not the same. However, for each $\varphi \in \mathcal{A}_\varphi$, we can choose $\varphi' \in \mathcal{A}'_{\varphi'}$ such that $\varphi' = \varphi$ in the exterior of Ω and $\varphi' = \boldsymbol{x} \cdot \boldsymbol{E}_0$ (possibly adding a constant) in $\delta\Omega$, i.e. $\boldsymbol{E}_0 - \nabla \cdot \varphi' = \mathbf{0}$ in $\delta\Omega$. Also, for each $\boldsymbol{A} \in \mathcal{A}_{\boldsymbol{A}}$, we can choose $\boldsymbol{A}' \in \mathcal{A}'_{\boldsymbol{A}'}$ such that $\boldsymbol{A}' = \boldsymbol{A}$ in the exterior of Ω and $\boldsymbol{A}' = \frac{1}{2}\boldsymbol{x} \times \boldsymbol{B}_0$ (possibly adding a constant vector) in $\delta\Omega$, i.e. $\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}' = \mathbf{0}$ in $\delta\Omega$. This construction can only be applied to the test functions for the direct functionals J_e and K_m and not the dual functionals K_e and J_m. Using the results from (A.8) and (A.20), we then have (for arbitrary $\varphi \in \mathcal{A}_\varphi$ and $\boldsymbol{A} \in \mathcal{A}_{\boldsymbol{A}}$)

$$J_\mathrm{e}(\varphi, \boldsymbol{E}_0) - J'_\mathrm{e}(\varphi', \boldsymbol{E}_0) = \int_{\mathbb{R}^3\backslash\Omega} (\boldsymbol{E}_0 - \nabla\varphi) \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}')(\boldsymbol{E}_0 - \nabla\varphi)\,\mathrm{d}V \tag{21}$$

$$K_\mathrm{m}(\boldsymbol{A}, \boldsymbol{B}_0) - K'_\mathrm{m}(\boldsymbol{A}', \boldsymbol{B}_0) = \int_{\mathbb{R}^3\backslash\Omega} (\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})(\boldsymbol{\mu}^{-1} - (\boldsymbol{\mu}')^{-1})(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})\,\mathrm{d}V. \tag{22}$$

Using the minimizing potentials φ_0 and $\boldsymbol{A}_0$ for the unprimed functionals, we obtain the inequalities, valid for any $\Omega' \subseteq \Omega$,

$$\epsilon_0 \boldsymbol{E}_0 \cdot (\gamma_\mathrm{e} - \gamma'_\mathrm{e})\boldsymbol{E}_0 \geqslant \int_{\mathbb{R}^3\backslash\Omega} (\boldsymbol{E}_0 - \nabla\varphi_0) \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}')(\boldsymbol{E}_0 - \nabla\varphi_0)\,\mathrm{d}V \tag{23}$$

$$\mu_0^{-1}\boldsymbol{B}_0 \cdot (-\gamma_\mathrm{m} + \gamma'_\mathrm{m})\boldsymbol{B}_0 \geqslant \int_{\mathbb{R}^3\backslash\Omega} (\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}_0)(\boldsymbol{\mu}^{-1} - (\boldsymbol{\mu}')^{-1})(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}_0)\,\mathrm{d}V. \tag{24}$$

In the following, an inequality between two matrices is understood as an inequality valid for all quadratic forms over the matrices, i.e. by $\boldsymbol{\epsilon} \geqslant \boldsymbol{\epsilon}'$ we mean $\boldsymbol{E}_0 \cdot \boldsymbol{\epsilon}\boldsymbol{E}_0 \geqslant \boldsymbol{E}_0 \cdot \boldsymbol{\epsilon}'\boldsymbol{E}_0$ for all $\boldsymbol{E}_0$. If $\boldsymbol{\epsilon} \geqslant \boldsymbol{\epsilon}'$ and $\boldsymbol{\mu}^{-1} \geqslant (\boldsymbol{\mu}')^{-1}$ (equivalent to $\boldsymbol{\mu}' \geqslant \boldsymbol{\mu}$), it is then seen that the right-hand sides of (23) and (24) are nonnegative, and hence we must have $\gamma_\mathrm{e} \geqslant \gamma'_\mathrm{e}$ if $\boldsymbol{\epsilon} \geqslant \boldsymbol{\epsilon}'$ and $\gamma'_\mathrm{m} \geqslant \gamma_\mathrm{m}$ if $\boldsymbol{\mu}' \geqslant \boldsymbol{\mu}$. This proves that both the electric and the magnetic polarizabilities are nondecreasing when the material parameters increase in the region exterior to Ω. If the material parameters are equal in the region exterior to both volumes, i.e. $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'$ and $\boldsymbol{\mu} = \boldsymbol{\mu}'$ in the region exterior to Ω, the right-hand sides of (23) and (24) are zero, and it is seen that $\gamma_\mathrm{e} \geqslant \gamma'_\mathrm{e}$ and $\gamma_\mathrm{m} \leqslant \gamma'_\mathrm{m}$ for arbitrary $\Omega' \subseteq \Omega$. Thus, when the region of PEC increases from Ω' to Ω, the electric polarizability is nondecreasing, but the magnetic polarizability is nonincreasing. A corresponding result is shown for isotropic dielectric bodies in [4].

When the structure consists of only PEC in vacuum, i.e. $\boldsymbol{\epsilon} = \epsilon_0$ and $\boldsymbol{\mu} = \mu_0$ everywhere, the minimum properties (13) and (16) imply

$$\epsilon_0 \boldsymbol{E}_0 \cdot \gamma_\mathrm{e}\boldsymbol{E}_0 = \epsilon_0 \int_{\mathbb{R}^3\backslash\Omega} |\nabla\varphi_0|^2\,\mathrm{d}V + \epsilon_0 V_\Omega |\boldsymbol{E}_0|^2 \tag{25}$$

$$-\mu_0^{-1}\boldsymbol{B}_0\cdot\gamma_\mathrm{m}\boldsymbol{B}_0=\mu_0^{-1}\int_{\mathbb{R}^3\setminus\Omega}|\nabla\times\boldsymbol{A}_0|^2\,\mathrm{d}V+\mu_0^{-1}V_\Omega|\boldsymbol{B}_0|^2. \tag{26}$$

Since the right-hand sides of these equations are positive, it is readily seen that the electric polarizability for PEC bodies in vacuum is positive, whereas the magnetic polarizability is negative. It is also seen that the amplitude of the polarizability in each case is always larger than the volume of the PEC body. When embedding PEC bodies in a magnetic material, there is an interplay where the PEC properties promote negative polarizability, whereas the material properties promote positive polarizability if $\boldsymbol{\mu}\geqslant\boldsymbol{\mu}_0$. A precise example is given by a PEC sphere of radius a, surrounded by a spherical layer with isotropic permeability μ and outer radius b. It can be shown that the total magnetic polarizability of this structure is zero if

$$\left(\frac{a}{b}\right)^3=\frac{2(\mu/\mu_0-1)}{2\mu/\mu_0+1}=1-\frac{3}{2\mu/\mu_0+1}. \tag{27}$$

If $(a/b)^3$ is larger than this value, the polarizability is negative, and if it is smaller, the polarizability is positive. In homogenization theory, inclusions with zero polarizability are called neutral [6, pp 134–9] and are typically constructed from layered spheres as this one.

5. Upper and lower bounds on the polarizabilities

Using the variational formulations, we can find upper and lower bounds for the polarizabilities by inserting any set of admissible trial potentials $(\varphi,\boldsymbol{F},\psi,\boldsymbol{A})$ in the inequalities

$$-K_\mathrm{e}(\boldsymbol{F},\boldsymbol{D}_0)\leqslant\epsilon_0\boldsymbol{E}_0\cdot\gamma_\mathrm{e}\boldsymbol{E}_0\leqslant J_\mathrm{e}(\varphi,\boldsymbol{E}_0) \tag{28}$$

$$-K_\mathrm{m}(\boldsymbol{A},\boldsymbol{B}_0)\leqslant\mu_0\boldsymbol{H}_0\cdot\gamma_\mathrm{m}\boldsymbol{H}_0\leqslant J_\mathrm{m}(\psi,\boldsymbol{H}_0), \tag{29}$$

where the applied fields are related by $\boldsymbol{D}_0=\epsilon_0\boldsymbol{E}_0$ and $\boldsymbol{B}_0=\mu_0\boldsymbol{H}_0$. Using for instance the finite element method (FEM) for solving the field equations, we can compute each functional and consider the numerical potentials as trial fields. Each set of numerical potentials $(\varphi_\mathrm{num},\boldsymbol{F}_\mathrm{num},\psi_\mathrm{num},\boldsymbol{A}_\mathrm{num})$ can then be inserted in inequalities (28) and (29), which provides a strict error bound for the numerical computation of the polarizabilities. A corresponding interpretation of variational bounds in homogenization theory can be found in [25].

When there are no PEC bodies, the zero potentials are admissible in inequalities (28) and (29), implying

$$\boldsymbol{D}_0\cdot\int_{\mathbb{R}^3}\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\mathrm{d}V\,\boldsymbol{D}_0\leqslant\epsilon_0\boldsymbol{E}_0\cdot\gamma_\mathrm{e}\boldsymbol{E}_0\leqslant\boldsymbol{E}_0\cdot\int_{\mathbb{R}^3}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\,\mathrm{d}V\,\boldsymbol{E}_0 \tag{30}$$

$$\boldsymbol{B}_0\cdot\int_{\mathbb{R}^3}\left(\boldsymbol{\mu}_0^{-1}-\boldsymbol{\mu}^{-1}\right)\mathrm{d}V\,\boldsymbol{B}_0\leqslant\mu_0\boldsymbol{H}_0\cdot\gamma_\mathrm{m}\boldsymbol{H}_0\leqslant\boldsymbol{H}_0\cdot\int_{\mathbb{R}^3}(\boldsymbol{\mu}-\boldsymbol{\mu}_0)\,\mathrm{d}V\,\boldsymbol{H}_0. \tag{31}$$

This states that the polarizabilities are bounded by the harmonic and arithmetic mean of the material parameters. In homogenization theory, this is known as the Wiener bounds [26].

From the monotonicity results in the previous section, it can be concluded that if we have a set of PEC regions included in each other, $\Omega'\subseteq\Omega\subseteq\Omega''$, then we have

$$\gamma_\mathrm{e}'\leqslant\gamma_\mathrm{e}\leqslant\gamma_\mathrm{e}'' \tag{32}$$

$$-\gamma_\mathrm{m}'\leqslant-\gamma_\mathrm{m}\leqslant-\gamma_\mathrm{m}'' \tag{33}$$

if the material parameters $\boldsymbol{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{\mu}(\boldsymbol{x})$ are identical in each case. If the polarizability can be computed for the regions Ω' and Ω'', this leads to bounds for the unprimed polarizability. For instance, for PEC spheres in vacuum, it is easy to show that

$$\gamma_\mathrm{e}=4\pi a^3\mathbf{I}=3V\mathbf{I},\qquad\gamma_\mathrm{m}=-2\pi a^3\mathbf{I}=-\tfrac{3}{2}V\mathbf{I}, \tag{34}$$

where V is the volume of the sphere. For an arbitrary PEC region Ω in vacuum, we can then formulate the bounds [21]

$$3V'\mathbf{I} \leqslant \gamma_{\mathrm{e}} \leqslant 3V''\mathbf{I} \tag{35}$$

$$3V'/2\mathbf{I} \leqslant -\gamma_{\mathrm{m}} \leqslant 3V''/2\mathbf{I}, \tag{36}$$

where V' is the volume of the largest sphere contained in the body and V'' is the volume of the smallest sphere containing the body. This result can be generalized to shapes such as ellipsoids.

6. Numerical example

To demonstrate the upper and lower bounds provided by the variational principles, we consider the case of a PEC sphere in vacuum. For an axially symmetric structure, we can reduce the problem to two dimensions using cylindrical coordinates, since the solution cannot depend on the azimuth angle ϕ. The vector potentials are also reduced to a single ϕ-component, which can be seen by considering the curl (where $r = \sqrt{x^2 + y^2}$ is the cylindrical radius)

$$\begin{aligned}\nabla \times \boldsymbol{F} &= \hat{\boldsymbol{r}}\left(\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) + \hat{\boldsymbol{z}}\left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\phi) - \frac{1}{r}\frac{\partial F_r}{\partial \phi}\right)\\ &= -\hat{\boldsymbol{r}}\frac{\partial F_\phi}{\partial z} + \hat{\boldsymbol{z}}\frac{1}{r}\frac{\partial}{\partial r}(rF_\phi),\end{aligned} \tag{37}$$

where we enforced the symmetry conditions that the potential should be independent of ϕ and the curl should not have a ϕ-component. The field equation in vacuum is $\nabla \times (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}) = \mathbf{0}$, and since $\boldsymbol{D}_0$ is constant we have

$$\nabla \times (\nabla \times \boldsymbol{F}) = -\hat{\boldsymbol{\phi}}\left[\frac{\partial^2 F_\phi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\phi)\right)\right] = \mathbf{0}. \tag{38}$$

The field equation for the scalar potential in vacuum is $\nabla \cdot (\boldsymbol{E}_0 - \nabla\varphi) = 0$, implying the ϕ-independent Laplace equation since $\boldsymbol{E}_0$ is constant:

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2} = 0. \tag{39}$$

The corresponding equations for the magnetic potentials $\boldsymbol{A}$ and ψ follow analogously. To summarize, the scalar and vector potentials satisfy the following differential equations and associated boundary conditions on the PEC surface (assuming that all exciting fields are directed along the z-direction)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2} = 0, \qquad \varphi = E_0 z \tag{40}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rF_\phi)}{\partial r}\right) + \frac{\partial^2 F_\phi}{\partial z^2} = 0, \qquad \hat{\boldsymbol{n}} \times \left[-\hat{\boldsymbol{r}}\frac{\partial F_\phi}{\partial z} + \hat{\boldsymbol{z}}\frac{1}{r}\frac{\partial}{\partial r}(rF_\phi)\right] = -D_0\hat{\boldsymbol{n}} \times \hat{\boldsymbol{z}} \tag{41}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2} = 0, \qquad \hat{\boldsymbol{n}} \cdot \left[\hat{\boldsymbol{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{z}}\frac{\partial\psi}{\partial z}\right] = H_0\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}} \tag{42}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\right) + \frac{\partial^2 A_\phi}{\partial z^2} = 0, \qquad A_\phi = -\frac{1}{2}B_0 r. \tag{43}$$

These equations are easily solved using finite element software such as Comsol Multiphysics, and we can compute the functionals J_{e}, K_{e}, J_{m} and K_{m} using different discretizations. Even

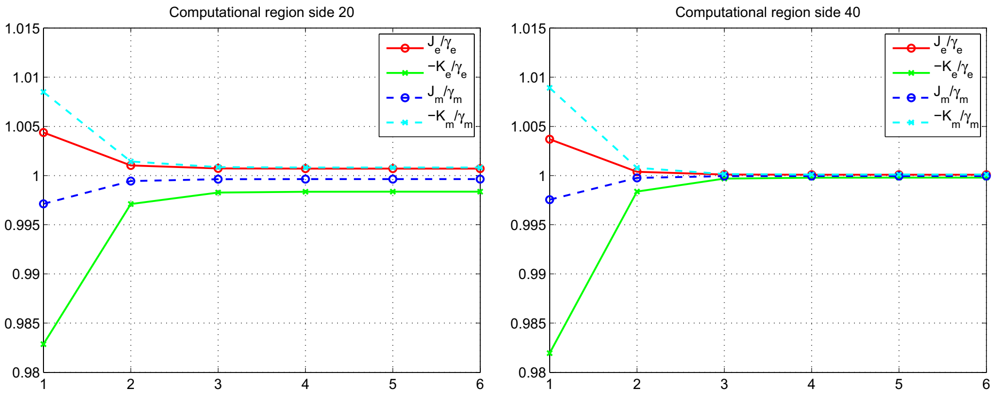

Figure 2. Demonstration of how numerical computations of the functionals provide bounds for γ_e and γ_m. The functionals are normalized by the exact values of γ_e and γ_m given by (34). Solid lines are for the electric case and dashed lines are for the magnetic case. The data points are for a PEC sphere with radius 1, centered in a computational cubic region with side 20 (left) or 40 (right). The horizontal scale corresponds to the discretization used in a 2D axial symmetric geometry (only orders of magnitude given, the numbers are slightly different for left and right figure): 200, 800, 3000, 12 000, 50 000 and 200 000 elements, respectively. The calculations are made with the commercial software Comsol Multiphysics 3.4 (http://www.comsol.com).

though the computations are necessarily performed in a finite computational domain D, the resulting numerical potentials are still admissible for an infinite domain if we require the fact that the potentials are zero on the boundary ∂D, and extend them to zero-valued functions on $\mathbb{R}^3 \setminus D$.

Each of the numerical computations provides a new bound for the polarizabilities, and in figure 2 we show how the bounds become progressively narrower as the discretization is made finer. The simulations are for a PEC sphere with radius 1, centered in a computational region in the shape of a cube with side 20 (left figure) or 40 (right figure). It is seen that the bounds level out after only a few refinements of the grid. The upper and lower bounds end up closer to each other for the larger computational region, which is due to the numerical potentials being better approximations to the free space potentials. In all cases, we have a strict bound on the polarizability.

When the bounds level out, the numerical solution has converged. In this respect, the only remaining error in the calculation of the polarizabilites is due to the modeling error in introducing a finite computational region. Thus, the variational principles also provide a means of determining the modeling error.

7. Conclusions

We have presented four variational principles from which the electric and magnetic polarizabilities can be computed or estimated. The polarizabilities are characterized as minima and maxima of these functionals, providing strict error bounds when applying numerical methods to compute the polarizabilities. Similar functionals have been presented before, and the purpose of this paper is to give a unified presentation of anisotropic permittivity and permeability in combination with PEC inclusions and to point out essential differences between electric and magnetic fields. The variational principles are valuable tools to estimate or bound the static polarizabilities, which showed up as important entities in recent work on physical limitations on the interaction between the electromagnetic field and linear, passive, causal structures such as antennas, materials and general scatterers [8–12].

The variational principles display important similarities and differences between electric and magnetic fields. If there are no PEC bodies present, there is a direct analogy between the functionals for the electric and magnetic cases, making them formally identical to each other. However, when a PEC body is introduced, the fields satisfy different boundary conditions on the PEC surface, which leads to different variational principles for the electric and magnetic cases, respectively. Specifically, the magnetic polarizability of a PEC body is negative, whereas the electric polarizability is positive. It is observed that in the electric case the boundary conditions are most easily expressed using a scalar potential, whereas the vector potential is most convenient in the magnetic case.

Appendix A. Verification of the variational principles for electric polarizability

In this appendix, we verify that the minimizing potentials for the direct and dual variational principles for the electric polarizability do indeed satisfy the electrostatic equations and that the minimum values correspond to the polarizability. The variational principles for the magnetic case can be shown in the same way.

Appendix A.1. Direct variational principle

The direct functional is

$$J_{\mathrm{e}}(\varphi, \boldsymbol{E}_0) = \int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi\cdot\boldsymbol{\epsilon}\nabla\varphi\,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega}\nabla\varphi\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V + \boldsymbol{E}_0\cdot\left[\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\,\mathrm{d}V + V_\Omega\boldsymbol{\epsilon}_0\right]\cdot\boldsymbol{E}_0, \tag{A.1}$$

where the admissible potentials satisfy the boundary condition $\varphi = \boldsymbol{E}_0\cdot\boldsymbol{x} + a$ on the PEC boundary, in order to satisfy $\hat{\boldsymbol{n}}\times(\boldsymbol{E}_0 - \nabla\varphi) = \boldsymbol{0}$. Now consider the variation of the functional at the minimum φ_0, ignoring terms quadratic and higher in $\delta\varphi$:

$$\begin{aligned}\frac{1}{2}\delta J_{\mathrm{e}} &= \frac{J_{\mathrm{e}}(\varphi_0+\delta\varphi, \boldsymbol{E}_0) - J_{\mathrm{e}}(\varphi_0, \boldsymbol{E}_0)}{2}\\ &= \int_{\mathbb{R}^3\setminus\Omega}\nabla\delta\varphi\cdot\boldsymbol{\epsilon}\nabla\varphi_0\,\mathrm{d}V - \int_{\mathbb{R}^3\setminus\Omega}\nabla\delta\varphi\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V\\ &= \int_{\mathbb{R}^3\setminus\Omega}\delta\varphi\nabla\cdot[\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi_0)]\,\mathrm{d}V + \oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot\delta\varphi\boldsymbol{\epsilon}(\boldsymbol{E}_0-\nabla\varphi_0)\,\mathrm{d}S\\ &\quad - \oint_{\partial\Omega}\delta\varphi\hat{\boldsymbol{n}}\cdot(\boldsymbol{\epsilon}_0\boldsymbol{E}_0)\,\mathrm{d}S.\end{aligned} \tag{A.2}$$

The last two integrals are identically zero since the variation $\delta\varphi$ must be zero on the PEC surface in order to comply with the boundary condition. Since the first variation of the functional should vanish at the extremum for all $\delta\varphi$, the minimizing potential must satisfy $\nabla\cdot[\boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi_0)] = 0$, i.e. the electrostatic equation.

We now show that the minimum value is indeed $\min_{\varphi\in\mathcal{A}_\varphi} J_{\mathrm{e}}(\varphi, \boldsymbol{E}_0) = J_{\mathrm{e}}(\varphi_0, \boldsymbol{E}_0) = \boldsymbol{E}_0\cdot\boldsymbol{p}$. The functional can be written as

$$J_{\mathrm{e}}(\varphi_0, \boldsymbol{E}_0) = \int_{\mathbb{R}^3\setminus\Omega}\nabla\varphi_0\cdot\boldsymbol{\epsilon}\nabla\varphi_0\,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega}\nabla\varphi_0\cdot(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\boldsymbol{E}_0\,\mathrm{d}V + \boldsymbol{E}_0\cdot\left[\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_0)\,\mathrm{d}V + V_\Omega\boldsymbol{\epsilon}_0\right]\cdot\boldsymbol{E}_0$$

$$= \int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi_0 \cdot \boldsymbol{\epsilon}(-\boldsymbol{E}_0 + \nabla\varphi_0)\,\mathrm{d}V + \int_{\mathbb{R}^3\setminus\Omega} \boldsymbol{E}_0 \cdot (\boldsymbol{\epsilon} - \epsilon_0)(\boldsymbol{E}_0 - \nabla\varphi_0)\,\mathrm{d}V + \int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi_0 \cdot \epsilon_0 \boldsymbol{E}_0\,\mathrm{d}V + \epsilon_0|\boldsymbol{E}_0|^2 V_\Omega. \tag{A.3}$$

Using integration by parts on the first integral implies

$$\int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi_0 \cdot \boldsymbol{\epsilon}(-\boldsymbol{E}_0 + \nabla\varphi_0)\,\mathrm{d}V = \int_{\mathbb{R}^3\setminus\Omega} \varphi_0 \nabla \cdot [\boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi_0)]\,\mathrm{d}V + \oint_{\partial\Omega} \varphi_0 \hat{\boldsymbol{n}} \cdot \boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi_0)\,\mathrm{d}V$$
$$= 0 + \oint_{\partial\Omega} (\boldsymbol{E}_0 \cdot \boldsymbol{x} + a)\hat{\boldsymbol{n}} \cdot \boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi_0)\,\mathrm{d}V = \boldsymbol{E}_0 \cdot \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}} \cdot \boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi_0)\,\mathrm{d}S, \tag{A.4}$$

where we used the field equation $\nabla \cdot [\boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi_0)] = 0$, the boundary condition $\varphi_0 = \boldsymbol{E}_0 \cdot \boldsymbol{x} + a$ on the PEC boundary and that the total charge on the PEC body is zero, i.e. $\oint_{\partial\Omega} \hat{\boldsymbol{n}} \cdot \boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi_0)\,\mathrm{d}S = 0$. In addition, we have

$$\int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi_0\,\mathrm{d}V = -\oint_{\partial\Omega} \hat{\boldsymbol{n}}\varphi_0\,\mathrm{d}S = -\oint_{\partial\Omega} \hat{\boldsymbol{n}}\boldsymbol{x}\,\mathrm{d}S \cdot \boldsymbol{E}_0 = -V_\Omega \boldsymbol{E}_0 \tag{A.5}$$

since $\oint_{\partial\Omega} \hat{\boldsymbol{n}}\boldsymbol{x}\,\mathrm{d}S = \int_\Omega \nabla\boldsymbol{x}\,\mathrm{d}V = V_\Omega \mathbf{I}$. Collecting the results, we find

$$J_\mathrm{e}(\varphi_0, \boldsymbol{E}_0) = \boldsymbol{E}_0 \cdot \left[\int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{\epsilon} - \epsilon_0)(\boldsymbol{E}_0 - \nabla\varphi_0)\,\mathrm{d}V + \oint_{\partial\Omega} \boldsymbol{x}\hat{\boldsymbol{n}} \cdot \boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi_0)\,\mathrm{d}S\right] = \boldsymbol{E}_0 \cdot \boldsymbol{p}. \tag{A.6}$$

We finally consider the difference between the two functionals for different geometries, primed and unprimed. We start by rewriting the functional as

$$J_\mathrm{e}(\varphi, \boldsymbol{E}_0) = \int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi \cdot \boldsymbol{\epsilon}\nabla\varphi\,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi \cdot (\boldsymbol{\epsilon} - \epsilon_0)\boldsymbol{E}_0\,\mathrm{d}V + \boldsymbol{E}_0 \cdot \left[\int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{\epsilon} - \epsilon_0)\,\mathrm{d}V + V_\Omega \epsilon_0\right]\boldsymbol{E}_0$$
$$= \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{E}_0 - \nabla\varphi) \cdot \boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi)\,\mathrm{d}V + 2\int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi \cdot \epsilon_0 \boldsymbol{E}_0\,\mathrm{d}V + \epsilon_0|\boldsymbol{E}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega} \mathrm{d}V + V_\Omega\right]$$
$$= \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{E}_0 - \nabla\varphi) \cdot \boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi)\,\mathrm{d}V + \epsilon_0|\boldsymbol{E}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega} \mathrm{d}V - V_\Omega\right], \tag{A.7}$$

where we used the fact that $\int_{\mathbb{R}^3\setminus\Omega} \nabla\varphi\,\mathrm{d}V = -V_\Omega \boldsymbol{E}_0$, as shown previously. Even though this expression involves infinite integrals, they cancel when looking at the difference between the two functionals. Assume that the PEC bodies are in regions Ω and Ω'. We then have

$$J_\mathrm{e}(\varphi, \boldsymbol{E}_0) - J'_\mathrm{e}(\varphi', \boldsymbol{E}_0) = \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{E}_0 - \nabla\varphi) \cdot \boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\varphi)\,\mathrm{d}V - \int_{\mathbb{R}^3\setminus\Omega'} (\boldsymbol{E}_0 - \nabla\varphi') \cdot \boldsymbol{\epsilon}'(\boldsymbol{E}_0 - \nabla\varphi')\,\mathrm{d}V. \tag{A.8}$$

Appendix A.2. Dual variational principle

The dual functional is

$$K_{\rm e}(\boldsymbol{F},\boldsymbol{D}_0)=\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}\nabla\times\boldsymbol{F}\,{\rm d}V-2\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F})\cdot\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\boldsymbol{D}_0\,{\rm d}V$$
$$-2\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})\,{\rm d}S$$
$$+\boldsymbol{D}_0\cdot\left[-\int_{\mathbb{R}^3\setminus\Omega}\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right){\rm d}V+V_\Omega\boldsymbol{\epsilon}_0^{-1}\right]\boldsymbol{D}_0,\tag{A.9}$$

where the admissible potentials satisfy the boundary condition $\hat{\boldsymbol{n}}\times\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})=\boldsymbol{0}$.

In the following, we need the identity

$$\oint_{\partial\Omega}\hat{\boldsymbol{n}}\times\boldsymbol{F}\,{\rm d}S=\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{F})\,{\rm d}V\tag{A.10}$$

for arbitrary fields $\boldsymbol{F}$. This is shown by the identity

$$(\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{F}))_i=x_i\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{F})=\hat{\boldsymbol{n}}\cdot(x_i\nabla\times\boldsymbol{F})=\hat{\boldsymbol{n}}\cdot(\nabla\times(x_i\boldsymbol{F})-\nabla x_i\times\boldsymbol{F})$$
$$=\hat{\boldsymbol{n}}\cdot(\nabla\times(x_i\boldsymbol{F}))-\hat{\boldsymbol{n}}\cdot(\hat{\boldsymbol{x}}_i\times\boldsymbol{F})=\hat{\boldsymbol{n}}\cdot(\nabla\times(x_i\boldsymbol{F}))+\hat{\boldsymbol{x}}_i\cdot(\hat{\boldsymbol{n}}\times\boldsymbol{F})\tag{A.11}$$

and since $\oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot(\nabla\times(x_i\boldsymbol{F}))\,{\rm d}S=-\int_{\mathbb{R}^3\setminus\Omega}\nabla\cdot(\nabla\times(x_i\boldsymbol{F}))\,{\rm d}V=0$, the identity (A.10) follows. When considering the dual magnetic functional $J_{\rm m}(\psi,\boldsymbol{H}_0)$, the corresponding integral identity is $\oint_{\partial\Omega}\hat{\boldsymbol{n}}\psi\,{\rm d}S=-\frac{1}{2}\oint_{\partial\Omega}\boldsymbol{x}\times(\hat{\boldsymbol{n}}\times\nabla\psi)\,{\rm d}S$ for arbitrary ψ, which is proven with similar techniques in [1]. We also make frequent use of the integration by parts formula (remember that the unit vector $\hat{\boldsymbol{n}}$ points out of the volume Ω)

$$\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{A})\cdot\boldsymbol{B}\,{\rm d}V=\int_{\mathbb{R}^3\setminus\Omega}\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})\,{\rm d}V-\oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot(\boldsymbol{A}\times\boldsymbol{B})\,{\rm d}S.\tag{A.12}$$

Now consider the variation of the functional at the minimum $\boldsymbol{F}_0$, ignoring terms quadratic and higher in $\delta\boldsymbol{F}$:

$$\frac{1}{2}\delta K_{\rm e}=\frac{K_{\rm e}(\boldsymbol{F}_0+\delta\boldsymbol{F},\boldsymbol{D}_0)-K_{\rm e}(\boldsymbol{F}_0,\boldsymbol{D}_0)}{2}=\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\delta\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}\nabla\times\boldsymbol{F}_0\,{\rm d}V$$
$$-\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\delta\boldsymbol{F})\cdot\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\boldsymbol{D}_0\,{\rm d}V-\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\nabla\times\delta\boldsymbol{F})\,{\rm d}S$$
$$=\int_{\mathbb{R}^3\setminus\Omega}\delta\boldsymbol{F}\cdot[\nabla\times(\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0))]\,{\rm d}V$$
$$-\oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot[\delta\boldsymbol{F}\times\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)]\,{\rm d}S-\int_{\mathbb{R}^3\setminus\Omega}\delta\boldsymbol{F}\cdot\left(\nabla\times\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0\right){\rm d}V$$
$$+\oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot\left(\delta\boldsymbol{F}\times\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0\right){\rm d}S-\oint_{\partial\Omega}\hat{\boldsymbol{n}}\times\delta\boldsymbol{F}\,{\rm d}S\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0.\tag{A.13}$$

The second integral is identically zero since $\hat{\boldsymbol{n}}\times\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F})=0$ on the PEC boundary. The third is zero since $\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0$ is constant and the last two integrals cancel each other. The only integral remaining is the first one, and since we should have $\delta K_{\rm e}=0$ for any $\delta\boldsymbol{F}$ at the extremum, we see that the minimizing potential must satisfy the equation $\nabla\times(\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0))=\boldsymbol{0}$, i.e. the electrostatic equation.

We now show that the minimum value is indeed $\min_{F\in\mathcal{A}_F} K_\mathrm{e}(\boldsymbol{F},\boldsymbol{D}_0) = K_\mathrm{e}(\boldsymbol{F}_0,\boldsymbol{D}_0) = -\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}p$. The functional can be written as

$$\begin{aligned}
K_\mathrm{e}(\boldsymbol{F}_0,\boldsymbol{D}_0) &= \int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F}_0)\cdot\boldsymbol{\epsilon}^{-1}\nabla\times\boldsymbol{F}_0\,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F}_0)\cdot\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\boldsymbol{D}_0\,\mathrm{d}V \\
&\quad - 2\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}S \\
&\quad + \boldsymbol{D}_0\cdot\left[-\int_{\mathbb{R}^3\setminus\Omega}\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\mathrm{d}V + V_\Omega\boldsymbol{\epsilon}_0^{-1}\right]\boldsymbol{D}_0 \\
&= \int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F}_0)\cdot\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}V \\
&\quad - \int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\cdot\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\boldsymbol{D}_0\,\mathrm{d}V \\
&\quad - \int_{\mathbb{R}^3\setminus\Omega}\nabla\times\boldsymbol{F}_0\,\mathrm{d}V\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0 \\
&\quad - 2\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}S + V_\Omega\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{D}_0.
\end{aligned} \tag{A.14}$$

The first integral is

$$\begin{aligned}
\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F}_0)\cdot\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}V &= \int_{\mathbb{R}^3\setminus\Omega}\boldsymbol{F}_0\cdot[\nabla\times(\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0))]\,\mathrm{d}V \\
&\quad - \oint_{\partial\Omega}\hat{\boldsymbol{n}}\cdot[\boldsymbol{F}_0\times\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)]\,\mathrm{d}S = 0,
\end{aligned} \tag{A.15}$$

which is zero due to the field equation $\nabla\times(\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)) = \boldsymbol{0}$ and the boundary condition $\hat{\boldsymbol{n}}\times(\boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)) = \boldsymbol{0}$. The third integral is

$$\begin{aligned}
-\int_{\partial\Omega}\nabla\times\boldsymbol{F}_0\,\mathrm{d}V &= \oint_{\partial\Omega}\hat{\boldsymbol{n}}\times\boldsymbol{F}_0\,\mathrm{d}S = \oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{F}_0)\,\mathrm{d}S \\
&= \oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(-\boldsymbol{D}_0+\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}S \\
&= -\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\,\mathrm{d}S\cdot\boldsymbol{D}_0 + \oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}S \\
&= -V_\Omega\boldsymbol{D}_0 + \oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}S.
\end{aligned} \tag{A.16}$$

Collecting the results, we find

$$\begin{aligned}
K_\mathrm{e}(\boldsymbol{F}_0,\boldsymbol{D}_0) &= -\int_{\mathbb{R}^3\setminus\Omega}(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\cdot\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\boldsymbol{D}_0\,\mathrm{d}V \\
&\quad - \boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}S = -\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{p}.
\end{aligned} \tag{A.17}$$

We finally consider the difference between the two functionals for different geometries, primed and unprimed. We start by rewriting the functional as (using $\boldsymbol{p}_\mathrm{pec} = \oint_{\partial\Omega}\boldsymbol{x}\hat{\boldsymbol{n}}\cdot(\boldsymbol{D}_0+\nabla\times\boldsymbol{F}_0)\,\mathrm{d}S$ for brevity)

$$\begin{aligned}
K_\mathrm{e}(\boldsymbol{F},\boldsymbol{D}_0) &= \int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F})\cdot\boldsymbol{\epsilon}^{-1}\nabla\times\boldsymbol{F}\,\mathrm{d}V - 2\int_{\mathbb{R}^3\setminus\Omega}(\nabla\times\boldsymbol{F})\cdot\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\boldsymbol{D}_0\,\mathrm{d}V \\
&\quad -2\boldsymbol{D}_0\cdot\boldsymbol{\epsilon}_0^{-1}\boldsymbol{p}_\mathrm{pec} + \boldsymbol{D}_0\cdot\left[-\int_{\mathbb{R}^3\setminus\Omega}\left(\boldsymbol{\epsilon}_0^{-1}-\boldsymbol{\epsilon}^{-1}\right)\mathrm{d}V + V_\Omega\boldsymbol{\epsilon}_0^{-1}\right]\cdot\boldsymbol{D}_0
\end{aligned}$$

$$= \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0 + \nabla \times \boldsymbol{F})\, \mathrm{d}V$$
$$-2\int_{\mathbb{R}^3\setminus\Omega} \nabla \times \boldsymbol{F}\, \mathrm{d}V \cdot \epsilon_0^{-1}\boldsymbol{D}_0 - 2\boldsymbol{D}_0 \cdot \epsilon_0^{-1}\boldsymbol{p}_{\mathrm{pec}} + \epsilon_0^{-1}|\boldsymbol{D}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega} \mathrm{d}V + V_\Omega\right]$$
$$= \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0 + \nabla \times \boldsymbol{F})\, \mathrm{d}V$$
$$+ \epsilon_0^{-1}|\boldsymbol{D}_0|^2\left[-\int_{\mathbb{R}^3\setminus\Omega} \mathrm{d}V - V_\Omega\right], \tag{A.18}$$

where we used that $\int_{\mathbb{R}^3\setminus\Omega} \nabla \times \boldsymbol{F}\, \mathrm{d}V = V_\Omega \boldsymbol{D}_0 - \boldsymbol{p}_{\mathrm{pec}}$, which was shown previously. The difference between the two functionals can then be written as

$$K_{\mathrm{e}}(\boldsymbol{F}, \boldsymbol{D}_0) - K'_{\mathrm{e}}(\boldsymbol{F}', \boldsymbol{D}_0) = \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}^{-1}(\boldsymbol{D}_0 + \nabla \times \boldsymbol{F})\, \mathrm{d}V$$
$$- \int_{\mathbb{R}^3\setminus\Omega'} (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}') \cdot (\boldsymbol{\epsilon}')^{-1}(\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}')\, \mathrm{d}V. \tag{A.19}$$

The results (A.8) and (A.19) are identical for the magnetic functionals $J_{\mathrm{m}}(\psi, \boldsymbol{H}_0)$ and $K_{\mathrm{m}}(\boldsymbol{A}, \boldsymbol{B}_0)$, respectively. For instance, we have

$$K_{\mathrm{m}}(\boldsymbol{A}, \boldsymbol{B}_0) - K'_{\mathrm{m}}(\boldsymbol{A}', \boldsymbol{B}_0) = \int_{\mathbb{R}^3\setminus\Omega} (\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}) \cdot \boldsymbol{\mu}(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A})\, \mathrm{d}V$$
$$- \int_{\mathbb{R}^3\setminus\Omega'} (\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}') \cdot \boldsymbol{\mu}'(\boldsymbol{B}_0 + \nabla \times \boldsymbol{A}')\, \mathrm{d}V. \tag{A.20}$$

References

[1] Kleinman R E 1973 Dipole moments and near field potentials *Appl. Sci. Res.* **27** 335–40

[2] Keller J B, Kleinman R E and Senior T B A 1972 Dipole moments in Rayleigh scattering *J. Inst. Math. Appl.* **9** 14–22

[3] Jones D S 1979 Low frequency electromagnetic radiation *J. Inst. Maths. Appl.* **23** 421–47

[4] Jones D S 1985 Scattering by inhomogeneous dielectric particles *Q. J. Mech. Appl. Math.* **38** 135–55

[5] Hashin Z and Shtrikman S 1962 A variational approach to the theory of the effective magnetic permeability of multiphase materials *J. Appl. Phys.* **33** 3125–31

[6] Milton G W 2002 *The Theory of Composites* (Cambridge: Cambridge University Press)

[7] Jikov V V, Kozlov S M and Oleinik O A 1994 *Homogenization of Differential Operators and Integral Functionals* (Berlin: Springer)

[8] Sohl C, Gustafsson M and Kristensson G 2007 Physical limitations on broadband scattering by heterogeneous obstacles *J. Phys. A: Math. Theor.* **40** 11165–82

[9] Gustafsson M, Sohl C and Kristensson G 2007 Physical limitations on antennas of arbitrary shape *Proc. R. Soc.* A **463** 2589–607

[10] Sohl C, Gustafsson M and Kristensson G 2007 Physical limitations on metamaterials: restrictions on scattering and absorption over a frequency interval *J. Phys. D: Appl. Phys.* **40** 7146–51

[11] Sohl C, Larsson C, Gustafsson M and Kristensson G 2008 A scattering and absorption identity for metamaterials: experimental results and comparison with theory *J. Appl. Phys.* **103** 054906

[12] Sohl C and Gustafsson M 2008 A priori estimates on the partial realized gain of ultra-wideband (UWB) antennas *Q. J. Mech. Appl. Math.* **61** 415–30

[13] Bossavit A 1988 Magnetostatic problems in multiply connected regions: some properties of the curl operator *IEE Proc.* A **135** 179–87

[14] Bossavit A 1988 Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism *IEE Proc.* A **135** 493–500

[15] Bossavit A 1996 Edge elements for magnetostatics *Int. J. Numer. Modelling: Electron. Netw. Devices Fields* **9** 19–34

[16] van Bladel J 1973 Circuit parameters from Maxwell's equations *Appl. Sci. Res.* **28** 381–97

[17] Kazantsev V P 1987 Variational principles of magnetostatics *Russ. Phys. J.* **30** 624–8

[18] Zolotov O A and Kazantsev V P 1991 Variational principles of magnetostatics of superconductors. I *Russ. Phys. J.* **34** 824–9

[19] Zolotov O A and Kazantsev V P 1991 Variational principles of magnetostatics of superconductors. II *Russ. Phys. J.* **34** 829–35

[20] Landau L D and Lifshitz E M 2004 *Fluid Mechanics (Course of Theoretical Physics* vol 6*)* 2nd edn (Oxford: Butterworth-Heinemann)

[21] Schiffer M and Szegö G 1949 Virtual mass and polarization *Trans. Am. Math. Soc.* **67** 130–205

[22] Payne L E 1967 Isoperimetric inequalities and their applications *SIAM Rev.* **9** 453–88

[23] Taylor G I 1928 The energy of a body moving in an infinite fluid, with an application to airships *Proc. R. Soc.* A **120** 13–21

[24] Cherkaev A V and Gibiansky L V 1994 Variational principles for complex conductivity, viscoelasticity, and similar problems in media with complex moduli *J. Math. Phys.* **35** 127–45

[25] Engström C and Sjöberg D 2007 On two numerical methods for homogenization of Maxwell's equations *J. Electromagn. Waves Appl.* **21** 1845–56

[26] Wiener O 1912 Die Theorie des Mischkörpers für das Feld des stationären Strömung *Abh. Math. Physichen Klasse Königl. Säcsh. Gesel. Wissen* **32** 509–604